PICARD GROUPS OF MODULI SPACE OF LOW DEGREE K3 SURFACES

ZHIYUAN LI, ZHIYU TIAN

Abstract. We study the moduli space of quasi-polarized K3 surfaces of degree 6 and 8 via geometric invariant theory. In particular, we verify the Noether-Lefschetz conjecture [24] in these two cases. The general case is discussed at the end of the paper.

1. Introduction

A primitively quasi-polarized K3 surface (S, L) of degree $2l$ consists of a K3 surfaces and a semiample line bundle L such that $c_1(L) \in H^2(S, \mathbb{Z})$ is a primitive class and $L^2 = 2l$. Let $\mathcal{M}_{2l}$ be the moduli space of primitively quasi-polarized $K3$ surfaces of degree $2l$. The Noether-Lefschetz divisors in $\mathcal{M}_{2l}$ correspond to $K3$ surfaces with Picard number at least 2.

More precisely, for any non-negative integers d, g, we define $D^{2l}_{d,g} \subset \mathcal{M}_{2l}$ to be the locus of quasi-polarized K3 surfaces $(S, L) \in \mathcal{M}_{2l}$ which contains a curve class $\beta \in \mathrm{Pic}(S)$ satisfying

$$\beta^2 = 2g - 2 \text{ and } \beta \cdot L = d.$$

In [24], Maulik and Pandharipande have conjectured that the Picard group with $\mathbb{Q}$-coefficients $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{M}_{2l})$ is spanned by those Noether-Lefschetz divisors $\{D^{2l}_{d,g}\}$ on $\mathcal{M}_{2l}$.

The case of $l = 1, 2$ can be deduced from [18], [31] and [17]. In the present paper, we study the birational models of $\mathcal{M}_6$ and $\mathcal{M}_8$ via geometric invariant theory and verify this conjecture. Our main result is:

Theorem 1.1. *All the Noether-Lefschetz divisors $\{D^{2l}_{d,g}\}$ are irreducible divisors on $\mathcal{M}_{2l}$. When $l = 3, 4$, the Picard group $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{M}_{2l})$ with rational coefficients is spanned by Noether-Lefschetz divisors $D^{2l}_{d,1}, d = 1, 2, 3, 4$.*

It is well-known that $\mathcal{M}_{2l}$ is a connected component of the Shimura variety of orthogonal type by global Torelli theorem. The Noether-Lefschetz conjecture is also closely related to the study of cohomology on such Shimura varieties. The vanishing of the first cohomology of $\mathcal{M}_{2l}$ is proved in [20], and actually we have the following result:

Theorem 1.2. *The Picard group $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{M}_{2l})$ is isomorphic to the cohomology group $H^2(\mathcal{M}_{2l}, \mathbb{Q})$ for any l.*

Outline of the paper. In section 2, we review the Noether-Lefshcetz (NL) divisors on $\mathcal{M}_{2l}$ from an arithmetic perspective and show that they are all irreducible divisors. The projective models of low degree K3 surfaces are described in section 3. In theses cases, we give precise geometry description of elements in certain NL divisors. Theorem 1.1 is proved in the section 4 and section 5 via geometric invariant theory (GIT). Roughly speaking, we can construct an open subset of $\mathcal{M}_{2l}$ via GIT and the boundary components are NL divisors. In the last section, we prove a more general result on arbitrary Shimura variety of orthogonal type and Theorem 1.2 is deduced as a corollary.

Acknowledgements. The authors are grateful to Brendan Hassett and Radu Laza for useful discussions.

2. Period space and Heegner divisors

2.1. **Period domain of K3 surface.** Let (S, L) be a primitively quasi-polarized K3 surface of degree $2l$. The middle cohomology $H^2(S, \mathbb{Z})$ is a unimodular even lattice of signature $(3, 19)$ under the intersection form $\langle , \rangle$. The orthogonal complement of the first Chern class $c_1(L)$ of L

$$\Lambda_{2l} := \langle c_1(L) \rangle^{\perp} \subset H^2(S, \mathbb{Z})$$

is an even lattice of signature $(2, 19)$, and it has a unique representation

$$\Lambda_{2l} = \mathbb{Z}\omega \oplus U^{\oplus 2} \oplus E_8(-1)^{\oplus 2}, \tag{2.1}$$

where $\langle \omega, \omega \rangle = -2l$, U is the hyperbolic plane and $E_8(-1)$ is the unimodular, negative definite even lattice of rank 8.

The period domain $\mathcal{D}_{2l}$ associated to Λ_{2l} can be realized as

$$\mathcal{D}_{2l} = \{v \in \mathbb{P}(\Lambda_{2l} \otimes_{\mathbb{Z}} \mathbb{C}) |\, \langle v, v \rangle = 0, -\langle v, \bar{v} \rangle > 0\}.$$

The arithmetic group

$$\Gamma_{2l} = \{g \in \mathrm{Aut}(\Lambda_{2l}) |\ g \text{ acts trivially on } \Lambda_{2l}^{\vee}/\Lambda_{2l}\},$$

naturally acts on $\mathcal{D}_{2l}$. According to the Global Torelli theorem of K3 surfaces, there is an isomorphism

$$\mathcal{M}_{2l} \cong \Gamma_{2l} \backslash \mathcal{D}_{2l}$$

via the period map. This implies that $\mathcal{M}_{2l}$ is a locally Hermitian symmetric variety. Moreover, $\mathcal{M}_{2l}$ is $\mathbb{Q}$-factorial since it only has quotient singularities.

2.2. **Heegner divisors.** Given an element $v \in \Lambda_{2l}^{\vee}$, there is an associated hyperplane

$$H_v := \{u \in \mathcal{D}_{2l} |\ \ \langle u, v \rangle = 0\} \subseteq \mathcal{D}_{2l}.$$

It is easy to see that the value $\langle v, v \rangle$ and the residue class of v modulo the lattice Λ_{2l} are both invariant under the action of Γ_{2l}. Thus, for each pair of

$n \in \mathbb{Q}^{<0}$ and $\gamma \in \Lambda_{2l}^{\vee}/\Lambda_{2l}$, one can define the *Heegner divisor* $y_{n,\gamma}$ of $\Gamma_{2l}\backslash\mathcal{D}_{2l}$ by

$$y_{n,\gamma} = \left(\bigcup_{\frac{1}{2}\langle v,v\rangle = n,\ v\equiv\gamma \bmod \Lambda_{2l}} H_v \right) /\Gamma_{2l}.$$

Using the identification $\mathcal{M}_{2l} \cong \Gamma_{2l}\backslash\mathcal{D}_{2l}$ via period map, Maulik and Pandharipande have showed that the Noether-Lefschetz divisors are exactly the Heegner divisors on $\Gamma_{2l}\backslash\mathcal{D}_{2l}$.

Lemma 2.3. [24] *The group $\Lambda_{2l}^{\vee}/\Lambda_{2l}$ is generated by the element $\frac{1}{2l}\omega$. The Noether-Lefschetz divisor $D_{d,g}^{2l} = y_{n,\gamma}$, where*

$$n = -\frac{\Delta_{d,g}}{4l},\ \textit{and}\ \gamma \equiv d(\frac{1}{2l}\omega) \mod \Lambda_{2l}.$$

Similarly as in [15], we prove the following theorem:

Theorem 2.4. *(***Irreducibility Theorem***) All the Heegner divisors $y_{n,\gamma}$ (or equivalently, Noether-Lefschetz divisors $D_{d,g}^{2l}$) are irreducible.*

Proof. Let $v \in \Lambda^{\vee}$ be a vector satisfying $\langle v, v\rangle = 2n$ and $v \equiv \frac{d\omega}{2l} \mod \Lambda_{2l}$. Denote by $k = \frac{2l}{(2l,d)}$, it corresponds to a primitive vector $v^{pr} \in \Lambda_{2l}$ with norm $N = 2nk^2$ and *level* k and *type* d. Here we say a primitive vector $u \in \Lambda_{2l}$ is of *level* k if $\langle u, \Lambda_{2l}\rangle = k\mathbb{Z}$ and it is of *type* d if $\frac{u}{k} \equiv \frac{d\omega}{2l} \mod \Lambda_{2l}$. Moreover, they satisfy that $\frac{N}{2k^2} + \frac{d^2}{4l}$ is an integer.

Obviously, we have $H_v = H_{v^{pr}}$. It is easy to see that for each hyperplane $H_{v^{pr}} \subset \mathcal{D}_{2l}$, the arithmetic quotient $H_{v^{pr}}/\Gamma_{2l}$ is irreducible. As the Heegner divisor $y_{n,\gamma}$ is a union of $H_{v^{pr}}/\Gamma_{2l}$ for all primitive vectors $v^{pr} \in \Lambda_{2l}$ with given norm, level and type, it suffices to prove that the arithmetic group Γ_{2l} acts transitively on all such primitive vectors in Λ_{2l}.

Let u_1, v_1 be the two generators of the hyperbolic plane $U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Next, we say that two elements in Λ_{2l} are congruent if they are in the same orbit under the action of Γ_{2l}. We claim that any primitive vector $v \in \Lambda_{2l}$ with norm N and level k and type d as above is congruent to the vector

$$\frac{dk}{2l}\omega + k(u_1 + mv_1), \tag{2.2}$$

where $m = \frac{N}{2k^2} + \frac{d^2}{4l} \in \mathbb{Z}$.

Let K be a rank one lattice of discriminant N. Each primitive vector in Λ_{2l} of norm N corresponds to a primitive imbedding $K \hookrightarrow \Lambda_{2l}$. Two primitive vectors are *congruent* if the corresponding imbedding differ by an automorphism in Γ_{2l}.

Now we use the Nikulin's theory [28] §1.15 on imbedding of quadratic forms to classify all the congruent classes of the primitive imbedding. According to [28], the primitive imbedding $K \hookrightarrow \Lambda_{2l}$ is uniquely determined by the data (H, H_K, ϕ, M), where

- H is a subgroup of $\Lambda_{2l}^\vee/\Lambda_{2l}$.
- H_K is a subgroup of $K^\vee/K$.
- An isomorphism $\phi : H_K \to H$ preserving the quadratic forms restricted to these subgroups, with graph $\Gamma_\phi \subseteq K^\vee/K \bigoplus \Lambda_{2l}^\vee/\Lambda_{2l}$.
- An even lattice M with signature $(2,18)$ and discriminant form q_M and an isomorphism $\phi_M : q_M \to -((q_K \oplus -q)|_{\Gamma_\phi^\perp})/\Gamma_\phi$. Here q_K is the discriminant quadratic form on $K^\vee/K$ and q the discriminant quadratic form on $\Lambda_{2l}^\vee/\Lambda_{2l}$.

Two imbeddings (H, H_K, ϕ, M) and (H', H_K', ϕ', M') are congruent if and only if $H_K = H_K'$ and $\phi = \phi'$.

In our situation, let v^{pr} be the image of the imbedding. Then the level of v^{pr} actually corresponds to the order of H which uniquely determines H since the discriminant group $\Lambda_{2l}/\Lambda_{2l}$ is a cyclic group of order $2l$. The isomorphism $\phi : H_K \to H$ corresponds to an automorphism of H which is uniquely determined by the type of v^{pr}. Hence the congruent class of the primitive imbedding can be classified by the level and type. Notice that the primitive vector (2.2) is of level k and type d. And we prove our claim. ♣

2.5. **Dimension formula.** Let $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{D}_{2l}/\Gamma_{2l})^{Heegner}$ be the subgroup of $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{D}_{2l}/\Gamma_{2k})$ generated by Heenger divisors with $\mathbb{Q}$-coefficients. By [7] and [24], the $\mathbb{Q}$-rank ρ_{2l} of $\mathrm{Pic}_{\mathbb{Q}}(\Gamma_{2l}\backslash\mathcal{D}_{2l})^{Heegner}$ can be explicitly computed by the following formula:

$$\begin{aligned}\rho_{2l} = {} & \frac{31}{24}l + \frac{55}{24} - \frac{1}{6\sqrt{6l}}\mathrm{Re}(e^{\frac{5\pi i}{12}}(G(-1,4l)+G(3,4l))) \\ & - \frac{1}{4\sqrt{2l}}\mathrm{Re}(G(-1,2l)) - \sum_{k=0}^{l}\{\frac{k^2}{4l}\} - \sharp\{k \mid \frac{k^2}{4l} \in \mathbb{Z}, 0 \le k \le l\}\end{aligned} \tag{2.3}$$

where $\{,\}$ denotes the fraction part and $G(a,b)$ is the generalized quadratic Gauss sum:

$$G(a,b) = \sum_{k=0}^{b-1} e^{2\pi i \frac{ak^2}{b}}.$$

Let us denote by $d_{Eis} = \sharp\{k \mid \frac{k^2}{4l} \in \mathbb{Z}, 0 \le k \le l\}$. After applying the summation formula proved by Gauss in 1811 (cf. [4] §2.2), one can simply get

Lemma 2.6.

$$\rho_{2l} = \frac{31l+55}{24} - \frac{1}{4}\alpha_l - \frac{1}{6}\beta_l - \sum_{k=0}^{l}\{\frac{k^2}{4l}\} - d_{Eis}, \tag{2.4}$$

where

$$\alpha_l = \begin{cases} 0, & l \textit{ is odd };\\ \left(\frac{2l}{2l-1}\right) & \textit{otherwise}.\end{cases}, \quad \beta_l = \begin{cases} \left(\frac{l}{4l-1}\right) - 1, & \textit{if } 3|l,\\ \left(\frac{l}{4l-1}\right) + \left(\frac{l}{3}\right) & \textit{otherwise}.\end{cases}$$

and $\left(\frac{a}{b}\right)$ *is the Jacobi symbol.*

In particular, we have $\rho_{2l} = 2, 3, 4, 4$, when $l = 1, 2, 3, 4$.

3. Projective models of K3 surfaces

Let S be a smooth K3 surface with a primitive quasi-polarization L satisfying $L^2 = 2l$ and $L \cdot C \geq 0$ for every curve $C \subset S$. The linear system $|L|$ defines a map ψ_L from S to $\mathbb{P}^{l+1}$. The image of ψ_L is called a *projective model* of S.

In [29], Saint-Donat gives a precise description of all projective models of (S, L) when ψ_L is not a birational morphism.

Proposition 3.1. [29] *Let L be the primitive quasi-polarization of degree $2l$ on S and let ψ_L be the map defined by $|L|$. Then there are following possibilities:*

1. *ψ_L is birational to a degree $2l$ surface in $\mathbb{P}^{l+1}$. In particular, ψ_L is a closed embedding when L is ample.*
2. *ψ_L is a generically $2:1$ map and $\psi_L(S)$ is a smooth rational normal scroll of degree l, or a cone over a rational normal curve of degree l.*
3. *$|L|$ has a fixed component D, which is a smooth rational curve. Moreover, $\psi_L(S)$ is a rational normal curve of degree $l+1$ in $\mathbb{P}^{l+1}$.*

We call K3 surfaces of type (1), (2), (3) *nonhyperelliptic*, *unigonal*, and *digonal* K3 surfaces accordingly. When $l = 2, 3, 4$, the projective model of a general quasi-polarized K3 surface (S, L) is a complete intersection in the projective space $\mathbb{P}^{l+1}$.

Remark 3.2. Assume that ψ_L is a birational morphism. Then one can easily see that L is not ample if and only if there exists an exceptional (-2) curve $D \subseteq S$. The morphism ψ_L will factor through a contraction $\pi : S \to \tilde{S}$ where $\tilde{S}$ is a singular K3 surface with A-D-E singularities.

Recalling that the Noether-Lefschetz divisor $D^{2l}_{0,0}$ parametrize all K3 surfaces (S, L) of degree $2l$ with exceptional (-2) curves. Therefore, the projective model of a general member in $D^{2l}_{0,0}$ is a surface in $\mathbb{P}^{l+1}$ of degree $2l$ with A-D-E singularities.

In this paper, we mainly consider the case $l = 3$ and 4, where the above classification can be easily read off from the Picard lattice of S.

Lemma 3.3. *Let (S, L) be a smooth quasi-polarized K3 surface of degree $2l$ ($l = 3, 4$). Then*

1. *$(S, L) \in D^{2l}_{1,1}$ if and only if S is digonal except*
   - ($*$) *$L^2 = 8$ and $L = L' + E + C$, where C is a rational curve, E is an irreducible elliptic curve and L' is irreducible of genus two with $L' \cdot C = E \cdot C = 1$ and $L' \cdot E = 2$. The image $\psi_L(S)$ is contained in a cone over cubic surface in $\mathbb{P}^4$.*
2. *$(S, L) \in D^{2l}_{2,1} \backslash D^{2l}_{1,1}$ if and only if S is unigonal.*

(3) $(S,L) \in D^{2l}_{3,1} \backslash (D^{2l}_{1,1} \cup D^{2l}_{2,1})$ *if and only if* S *is one of the following:*
- *when* $l = 3$, S *is birational to the complete intersection of a singular quadric and a cubic in* $\mathbb{P}^4$ *via* ψ_L.
- *when* $l = 4$, S *is either birational to a bidegree* $(2,3)$ *hypersurface of the Serge variety* $\mathbb{P}^1 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^5$ *via* ψ_L *or is in case* $(*)$.

Proof. The proof of (1) and (2) are straightforward from Proposition 3.1. See also [29] §2, §5 for more detailed discussion.

Now we suppose that a quasi-polarized K3 surface $(S,L) \in D^6_{3,1}$ is neither unigonal or diagonal. Then ψ_L is a birational map to a complete intersection of a quadric and a cubic. Our first statement of (3) comes from the fact any quadric threefold containing a plane cubic must be singular. If $(S,L) \in D^8_{3,1}$, the assertion follows from [29] Proposition 7.15 and Example 7.19. ♣

Remark 3.4. We also refer the readers to David Morrison's lecture notes [10] for a similar discussion and [16] for a complete classification of all projective models of low degree K3 surfaces (e.g. Mukai models).

4. Complete intersection of a quadric and a cubic

In this section, we construct the moduli space of the complete intersection of a smooth quadric and a cubic in $\mathbb{P}^4$ via geometric invariant theory.

4.1. **Terminology and Notations.** In the rest of this paper, we will use the following terminology. Let $f(u,v,w)$ be an analytic function in $\mathbb{C}[[u,v,w]]$ whose leading term defines an isolated singularity at the origin. We have the following types of singularities:

- Simple singualrities: isolated A_n, D_k, E_r singularities.
- Simple elliptic singularities $\tilde{E}_r$:
  - $\tilde{E}_6$: $f = u^3 + v^3 + w^3 + auvw$,
  - $\tilde{E}_7$: $f = u^2 + v^4 + w^4 + auvw$,
  - $\tilde{E}_8$: $f = u^2 + v^3 + w^6 + auvw$,

We will use the notation $l(x), q(x), c(x)$ as linear, quadratic and cubic polynomials of $x = (x_0, \ldots, x_n)$.

4.2. **Cubic sections on quadric threefolds.** Let Q be the smooth quadric threefold in $\mathbb{P}^4$ defined by the equation

$$x_0x_4 + x_1x_3 + x_2^2 = 0.$$

Since every nonsingular quadric hypersurface in $\mathbb{P}^4$ is projectively equivalent to Q, a complete intersection of a smooth quadric and a cubic can be identified with an element in $|\mathcal{O}_Q(3)|$.

The automorphism group of Q is the reductive Lie group $SO(Q)(\mathbb{C})$ which is isomorphic to $SO(5)(\mathbb{C})$. Then we can naturally describe the moduli space of the complete intersection of a smooth quadric and a cubic as the GIT

quotient of the linear system $|\mathcal{O}_Q(3)| = \mathbb{P}(V)$, where V is a 30-dimensional vector space defined by the exact sequence

$$0 \to H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(1)) \to H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(3)) \to V \to 0.$$

Let us take the set of monomials

$$\mathcal{B} := \{x_0^{a_0} x_1^{a_1} \dots x_4^{a_4} | \sum_{i=0}^{4} a_i = 3 \text{ and } a_0 a_4 = 0\}. \tag{4.1}$$

to be a basis of V. Sometimes, we may change the basis for simpler computations.

4.3. **Numerical criterion.** Now we classify stability of the points in $\mathbb{P}(V)$ under the action of $SO(Q)(\mathbb{C})$ by applying the Hilbert-Mumford numerical criterion [27].

As is customary, a one parameter subgroup (1-PS) of $SO(Q)(\mathbb{C})$ can be diagonalized as

$$\lambda_{u,v} : t \in \mathbb{C}^* \to \mathrm{diag}(t^u, t^v, 1, t^{-v}, t^{-u}),$$

for some $u, v \in \mathbb{Z}$. We call such $\lambda_{u,v} : \mathbb{C}^* \to SO(Q)(\mathbb{C})$ a *normalized* 1-PS of $SO(Q)(\mathbb{C})$ if $u \geq v \geq 0$.

Let $\lambda_{u,v}$ be a normalized 1-PS of $SO(Q)(\mathbb{C})$. Then the weight of a monomial $x_0^{a_0} x_1^{a_1} \dots x_4^{a_4} \in \mathcal{B}$ with respect to $\lambda_{u,v}$ is

$$(a_0 - a_4)u + (a_1 - a_3)v. \tag{4.2}$$

If we denote by $M_{\leq 0}(\lambda_{u,v})$ (resp. $M_{<0}(\lambda_{u,v})$) the set of monomials of degree 3 which have non-positive (resp. negative) weight with respect to $\lambda_{u,v}$, one can easily compute the maximal subsets $M_{\leq 0}(\lambda_{u,v})$ (resp. $M_{<0}(\lambda_{u,v})$), as listed in Table 1 (resp. Table 2) .

TABLE 1. Maximal subsets $M_{\leq 0}(\lambda)$

Cases	(u, v)	Maximal monomials
(N1)	(1,0)	$x_1^{a_1} x_2^{a_2} x_3^{a_3}, \sum a_i = 3$
(N2)	(1,1)	$x_0x_2x_3, x_1x_2x_3, x_1x_2x_4, x_2^3$
(N3)	(2,1)	$x_0x_3^2, x_1^2x_4, x_1x_2x_3, x_2^3$

TABLE 2. Maximal subsets $M_{<0}(\lambda)$

Cases	(u, v)	Maximal monomials
(U1)	(1,0)	$x_1^2x_4$
(U2)	(1,1)	$x_0x_3^2, x_2^2x_3$

According to the Hilbert-Mumford criterion, an element $f(x_0, \dots, x_4) \in \mathbb{P}(V)$ is not properly stable (resp. unstable) if and only if the weight of some monomial in f is non-positive (resp. negative). Thus we obtain:

Lemma 4.4. *Let X be the surface defined by an element in $\mathbb{P}(V)$. Then X is not properly stable if and only if $X = Q \cap Y$ for some cubic hypersurface $Y \subseteq \mathbb{P}^4$ defined by a cubic polynomial in one of following cases:*

- $c(x_1, x_2, x_3, x_4)$;
- $x_0x_3l(x_2, x_3) + x_1x_2l_1(x_3, x_4) + x_1q(x_3, x_4) + c(x_2, x_3, x_4)$;
- $x_0x_3^2 + x_1x_3l_1(x_2, x_3) + x_1x_4l_2(x_1, x_2, x_3) + c(x_2, x_3, x_4)$.

For $f \in \mathbb{P}(V)$ not properly stable, using the destabilizing 1-PS λ, the limit $\lim_{t\to 0} f_t = f_0$ exists and it is invariant with respect to λ. The invariant part of polynomials of type $(N1) - (N3)$ are the followings:

(α) $c(x_1, x_2, x_3) = 0$;
(β) $\lambda_1x_2^3 + \lambda_2x_1x_2x_3 + \lambda_3x_0x_2x_3 + \lambda_4x_1x_2x_4 = 0$, $\lambda_i \in \mathbb{C}$;
(γ) $\lambda_1x_2^3 + \lambda_2x_1x_2x_3 + \lambda_3x_0x_3^2 + \lambda_4x_1^2x_4 = 0$, $\lambda_i \in \mathbb{C}$.

Similarly, we get

Lemma 4.5. *With the notation above, X is not semistable if and only if $X = Q \cap Y$ for some cubic hypersurface Y defined by one of the following equations:*

- $x_4q(x_1, x_2, x_3, x_4)$;
- $x_0x_3^2 + x_1q(x_3, x_4) + c(x_2, x_3, x_4)$, *and* $c(x_2, x_3, x_4)$ *has no* x_2^3 *term.*

4.6. **Geometric interpretation of stability.** We use the terminology of the corank of the hypersurface singularities as in [1] and [21].

Definition 1. Let $0 \in \mathbb{C}^n$ be a hypersurface singularity given by an equation $f(z_1, \ldots, z_n) = 0$. The corank of 0 is n minus the rank of the Hessian of $f(z_1, \ldots, z_n)$ at 0.

Theorem 4.7. *A complete intersection $X = Q \cap Y$ is not properly stable if and only if X satisfies one of the following conditions:*

(i) X has a hypersurface singularity of corank 3.
(ii) X is singular along a line L and there exists a plane P such that $P \cap Q = 2L$ and P is contained in the projective tangent cone $\mathbb{P}(CT_p(X))$ for any point $p \in L$.
(iii) X has a singularity p of corank at least 2 and the restriction of the projective cone $\mathbb{P}(CT_p(X))$ to X contains a line L passing through p with multiplicity at least 6.

Proof. As a consequence of Lemma 4.4, it suffices to find the geometric characterizations of the complete intersections of type $(N1) - (N3)$. Here we do it case by case.

(i). If X is of type $(N1)$, then X can be considered as the intersection of Q and a cubic cone Y with the vertex $p_0 = [1, 0, 0, 0, 0] \in Q$. It is easy to see that p_0 is a corank of 3 singularity of X.

Conversely, we write the equation of Y as

$$x_0q(x_0, x_1, x_2, x_3) + c(x_1, x_2, x_3, x_4) = 0.$$

If we choose the affine coordinate

$$y_i := x_i/x_0, \tag{4.3}$$

then the affine equation near p_0 is

$$q(1, y_1, y_2, y_3) + c(y_1, y_2, y_3, -y_2^2 - y_1 y_3) = 0. \tag{4.4}$$

in $\mathbb{C}^3$. It has a corank 3 singularity at the origin if and only if the quadric q is 0.

(ii). If X is of type $(N2)$, then the equation of Y is given by

$$x_0 x_3 l(x_2, x_3) + x_1 x_2 l_1(x_3, x_4) + x_1 q(x_3, x_4) + c(x_2, x_3, x_4),$$

and therefore X is singular along the line $L : x_2 = x_3 = x_4 = 0$.

Moreover, for any point $p = [z_0, z_1, 0, 0, 0] \in L$, the projective tangent cone $\mathbb{P}(CT_p(X))$ at p is defined as

$$z_0 x_4 + z_1 x_3 = z_0 x_3 l(x_2, x_3) + z_1 q(x_2, x_3, x_4) = 0, \tag{4.5}$$

which contains the plane $P : x_3 = x_4 = 0$ for each $p \in L$ and $P \cap Q = 2L$.

Conversely, since the intersection of P and Q is a double line L, we may certainly assume that the plane P is defined by

$$x_3 = x_4 = 0$$

after some coordinate transform persevering the quadric form Q. Then the line $L = P \cap Q$ is given by $x_2 = x_3 = x_4 = 0$.

Because X is singular along L, the equation of Y can be written as:

$$x_0 q_1(x_2, x_3) + x_1 q_2(x_2, x_3, x_4) + c(x_2, x_3, x_4) = 0. \tag{4.6}$$

Then the projective tangent cone

$$\mathbb{P}(CT_p(X)) = \{z_0 x_4 + z_1 x_3 = z_0 q_1(x_2, x_3, x_4) + z_1 q_2(x_2, x_3, x_4) = 0\}$$

contains the plane P for each point $p = [z_0, z_1, 0, 0, 0] \in L$ only if the quadrics q_i have no x_2^2 term.

(iii). For X of type (N_3), a similar discussion is as follows: if Y is defined by

$$x_0 x_3^2 + x_1 x_3 l_1(x_2, x_3) + x_1 x_4 l_2(x_1, x_2, x_3) + c(x_2, x_3, x_4) = 0, \tag{4.7}$$

then $X = Q \cap Y$ is singular at p_0. After choosing the affine coordinates as (4.3), the affine equation near p_0 is

$$y_3^2 + y_1 y_3 f(y_1, y_2, y_3) + g(y_2, y_3) = 0 \tag{4.8}$$

for some polynomials f, g with $deg(f) \geq 1, deg(g) \geq 3$. Therefore, p_0 is a hypersurface singularity of corank 2 and its projective tangent cone is a double plane $2P : x_3^2 = x_4 = 0$. The remaining part is straightforward.

Conversely, we take p_0 to be the singular point as before. Then the equation of Y can be written as

$$x_0 q_1(x_1, \dots, x_3) + x_1 q_2(x_1, \dots, x_4) + c(x_2, x_3, x_4) = 0.$$

Then the quadric $q_1(x_1, x_2, x_3)$ is of the form $l(x_1, x_2, x_3)^2$ for some linear polynomial l because p_0 is singular of corank at least 2.

After we make a coordinate change preserving Q and p_0, the equation of Y can be written as either

$$x_0x_3^2 + x_1q(x_1, x_2, x_3, x_4) + c(x_2, x_3, x_4) = 0, \tag{4.9}$$

or

$$x_0x_2^2 + x_1q(x_1, x_2, x_3, x_4) + c(x_2, x_3, x_4) = 0. \tag{4.10}$$

The projective tangent cone at $\mathbb{P}(CT_{p_0}(X))$ is a double plane

$$2P : x_4 = x_3^2 = 0, \text{ or } x_4 = x_2^2 = 0.$$

The line L contained in the restriction of $2P$ to X has to be defined by $x_2 = x_3 = x_4 = 0$. It follows that the last case (4.10) can not happen since $P \cap X$ contains L with multiplicity at least 3.

Finally, the multiplicity condition implies that the quadric $q(x_1, x_2, x_3, x_4)$ does not have x_1^2, x_1x_2, x_2^2 terms. ♣

Remark 4.8. In the case of (N3), if we set $y_1 = w, y_2 = v$ and $y_3 = u$, then we see from (4.8) that the local analytic function near p_0 is equivalent to

$$u^2 + v^3 + w^6 + auvw = 0$$

in $\mathbb{C}[[u, v, w]]$. So a general member X of type (N_3) will have an isolated simple elliptic singularity of type $\tilde{E}_8$.

Theorem 4.9. *A complete intersection $X = Q \cap Y$ is unstable if and only if X satisfies one of the following conditions:*

(i′) $X = X_1 \cup X_2$ is reducible, where X_1 is a cone over a conic with vertex p and X_2 is singular at p;

(ii′) X is singular along a line L satisfying the condition: there exist a plane P such that $\mathbb{P}(CT_p(X)) = 2P$ for any point $p \in L$.

Proof. It suffices to check the complete intersections of type $(U1) - (U2)$ case by case.

(i'). Suppose $X = X_1 \cup X_2$ is a union of two surfaces satisfying the desired conditions. We can also assume that the vertex of X_1 is p_0 and X_1 is defined by

$$x_4 = x_2^2 + x_1x_3 = 0,$$

for a suitable change of coordinates preserving Q. Therefore, the equation of Y has the form

$$x_4q(x_0, \ldots, x_4) = 0.$$

Since the other component $X_2 : q(x_0, \ldots, x_4) = x_0x_4 + x_1x_3 + x_2^2$ is singular at p_0, there is no x_0x_i terms in the quadric $q(x_0, \ldots, q_4)$. The converse is obvious.

(ii'). To simplify the proof, we choose another monomial basis of V as below:

$$\mathcal{B}' := \{x_0^{a_0} \dots x_4^{a_4} | \sum_{i=0}^{4} a_i = 3,\ a_2 \leq 1\}. \tag{4.11}$$

Then the polynomial of type $(U2)$ has the form

$$x_0q_0(x_3, x_4) + x_1q_1(x_3, x_4) + x_2q_2(x_3, x_4) + c(x_3, x_4) = 0. \tag{4.12}$$

At this time, X is singular along the line $L : x_2 = x_3 = x_4 = 0$ and satisfies the condition described in (ii').

On the other hand, the line L on Q can be written as

$$L : x_2 = x_3 = x_4 = 0$$

for a suitable change of coordinates preserving Q. Then the equation of Y has the form

$$\sum_{i=0}^{1} x_iq_i(x_2, x_3, x_4) + x_2q_2(x_3, x_4) + c(x_3, x_4) = 0,$$

where q_i does not contain x_2^2 term.

Moreover, for any point $p = [z_0, z_1, 0, 0, 0] \in L$, the projective tangent cone $\mathbb{P}(CT_p(X))$ is given by

$$z_0x_3 + z_1x_4 = z_0q_0(x_2, x_3, x_4) + z_1q_1(x_2, x_3, x_4) = 0.$$

They have a common plane P with multiplicity 2 if and only if P is defined by $x_3 = x_4 = 0$ and $q_i(x_2, x_3, x_4)$ does not contain the x_2x_3, x_2x_4 terms. ♣

Corollary 4.10. *A complete intersection $X = Q \cap Y$ is semistable (reps. stable) if X has at worst isolated singularities (reps. simple singularities).*

Proof. By Theorem 4.9, the singular locus of X is at least one dimensional if it is unstable. Then X has to be semistable if it has at worst isolated singularities.

Next, from Theorem 4.7 and Remark 4.8, we know that if X is not properly stable, then either X is singular along a curve or it contains at least an isolated simple elliptic singularity. It follows that X with simple singularities is stable. ♣

Now it makes sense to talk about the moduli space $\mathcal{K}_6$ of complete intersections of a smooth quadric and a cubic with simple singularities. Let $\mathcal{U}_6$ be the open subset of $\mathbb{P}(V)^s$ parameterizing such complete intersections in $\mathbb{P}^4$. Then we have $\mathcal{K}_6 = \mathcal{U}_6//SO(5)(\mathbb{C})$.

Theorem 4.11. *There is a natural open immersion $\mathcal{P}_6 : \mathcal{K}_6 \to \mathcal{M}_6$, and the complement of the image $\mathcal{P}_6$ in $\mathcal{M}_6$ is the union of three Noether-Lefschetz divisors $D^6_{1,1}, D^6_{2,1}$ and $D^6_{3,1}$. In particular, the Picard group $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{M}_6)$ is spanned by $\{D^6_{d,1},\ 1 \leq d \leq 4\}$.*

Proof. For the first statement, one only need the fact that the complete intersections with simple singularities correspond to degree 6 quasi-polarized K3 surfaces containing a (-2) curve. Therefore, we obtain an open immersion $\mathcal{P}_6 : \mathcal{K}_6 \to \mathcal{M}_6$. By Lemma 3.3, we know that the boundary divisors of the image $\mathcal{P}_6(\mathcal{K}_6)$ is the union of $D^6_{1,1}, D^6_{2,1}$ and $D^6_{3,1}$.

Next, we claim that the dimension of $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{K}_6)$ is at most one. Observe that $\mathcal{K}_6$ is constructed via the GIT quotient $\mathcal{U}_6//SO(5)(\mathbb{C})$, and $\mathrm{Pic}(\mathcal{U}_6) \cong \mathrm{Pic}(\mathbb{P}(W))$ has rank one since the boundary of $\mathcal{U}_6$ in $\mathbb{P}(W)$ has codimension at least two. Let $\mathrm{Pic}(\mathcal{U}_6)_{SO(5)(\mathbb{C})}$ be the set of $SO(5)(\mathbb{C})$-linearized line bundles on $\mathcal{U}_6$. There is an injection

$$\mathrm{Pic}(\mathcal{U}_6//SO(5)(\mathbb{C})) \hookrightarrow \mathrm{Pic}(\mathcal{U}_6)_{SO(5)(\mathbb{C})}$$

by [19] Proposition 4.2. Our assertion follows from the fact the forgetful map $\mathrm{Pic}(\mathcal{U}_6)_{SO(5)(\mathbb{C})} \to \mathrm{Pic}(\mathcal{U}_6)$ is an injection.

Since the complement of $\mathcal{K}_6$ in $\mathcal{M}_6$ is the union of three irreducible divisors and $\dim_{\mathbb{Q}}(\mathrm{Pic}(\mathcal{K}_6)) \geq 4$, it follows that $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{M}_{2l})$ is spanned by the set of Noether-Lefschetz divisors $\{D^6_{d,1}, 1 \leq d \leq 4\}$ by dimension considerations. ♣

Remark 4.12. There is another natural GIT construction of moduli space of complete intersections in projective space, see [2]. There exists a projective bundle $\pi : \mathbb{P}E \to \mathbb{P}(H^0(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(2))) \cong \mathbb{P}^{14}$ parameterizing all complete intersections of a quadric and a cubic in $\mathbb{P}^5$. Then one can consider the GIT quotient

$$\mathbb{P}(E)//_{H_t} SL_5(\mathbb{C})$$

for the line bundle $H_t = \pi^*\mathcal{O}_{\mathbb{P}^{14}}(1) + t\mathcal{O}_{\mathbb{P}E}(1)$.

We want to point out that $\mathbb{P}(E)//_{H_t} SL_5(\mathbb{C})$ is isomorphic to our GIT quotient $\mathbb{P}(V)//SO(5)(\mathbb{C})$ when $t < 1/6$. This can be obtained via a similar argument as in [8]. It will be interesting to study the variation of GIT on $\mathbb{P}(E)//_{H_t} SL_5(\mathbb{C})$.

4.13. **Minimal orbits.** In this subsection, we give a description of the boundary components of the GIT compactification. The boundary of the GIT compactification consists of strictly semistable points with minimal orbits. From §3.2, it suffices to discuss the points of type $(\alpha)-(\gamma)$. As in [21], our approach is to use Luna's criterion:

Lemma 4.14. *(Luna's criterion)*[23] *Let G be a reductive group acting on an affine variety V. If H is a reductive subgroup of G and $x \in V$ is stabilized by H, then the orbit $G \cdot x$ is closed if and only if $C_G(H) \cdot x$ is closed.*

To start with, we first observe that Type (α), (β) and (γ) have a common specialization, which we denote by Type (ξ):

$$\lambda_1 x_2^3 + \lambda_2 x_1 x_2 x_3 = 0.$$

Lemma 4.15. *If X is of Type (ξ), it is strictly semistable with closed orbits.*

Proof. The stabilizer of Type (ξ) contains a 1-PS:

$$H = \{diag(t^2, t, 1, t^{-1}, t^{-2})|\ t \in \mathbb{C}^*\},$$

of distinct weights. So the center

$$C_G(H) = \{diag(a_0, a_1, 1, a_1^{-1}, a_0^{-1})\} \subset SO(Q)(\mathbb{C})$$

is a maximal torus. It acts on $V^H = \langle x_0x_3^2, x_1^2x_4, x_1x_2x_3, x_2^3\rangle \subset V$. It is straightforward to see any element of Type (ξ) is semistable with closed orbit in V^H under the action. Then the statement follows from Luna's criterion. ♣

Proposition 4.16. *Let X be a surface of Type (α). Then it has two corank 3 singularities. Moreover, we have*

(1) X is unstable if it is union of a quadric surface and a quadric cone with multiplicity two.
(2) The orbit of X is not closed if X is singular along two lines. It degenerates to type ξ.

Otherwise, X is semistable with closed orbit.

Proof. The stabilizer of Type (α) contains a 1-PS:

$$H_1 = \{diag(t, 1, 1, 1, t^{-1})|\ t \in \mathbb{C}^*\}.$$

The center $C_G(H_1) \cong SO(Q_1)(\mathbb{C}) \times SO(Q_2)(\mathbb{C})$, where $Q_1 = x_0x_4$ and $Q_2 = x_1x_3 + x_2^2$. The group $SO(Q_1)(\mathbb{C}) \cong SO(2;\mathbb{C})$ acts linearly on variable x_0, x_4, while $SO(Q_2)(\mathbb{C}) \cong SO(3)(\mathbb{C})$ acts linearly on the variables x_1, x_2 and x_3.

The action of $C_G(H_1)$ on $V^{H_1} = \left\langle x_1^{d_1}x_2^{d_2}x_3^{d_3}, \sum_{k=1}^{3} d_k = 3\right\rangle \subset V$ is equivalent to the action of $SO(Q_2)(\mathbb{C})$ on the set of cubic polynomials in three variables x_1, x_2, x_3 preserving the quadratic form Q_2. By Luna's criterion, we can reduce our problem to an simpler GIT question $V^{H_1}//SO(3)(\mathbb{C})$.

Any 1-PS $\lambda : \mathbb{C}^* \to SO(Q_2)(\mathbb{C})$ of $SO(Q_2)(\mathbb{C})$ can be diagonalized in the form

$$\lambda(t) = diag(t^a, 1, t^{-a}). \tag{4.13}$$

The weight of a monomial $x_1^{d_1}x_2^{d_2}x_3^{d_3}$ with respect to (4.13) is $a(d_1 - d_2)$. Then our assertion follows easily from the Hilbert-Mumford criterion. ♣

The remaining cases can be shown in a similar way. Here we omit the proof.

Proposition 4.17. *Let X be a surface of type (β). Then it is a union of a quadric surface and a complete intersection of two quadrics. Moreover, we have*

(i) X is unstable if X consists of two quadric cones and a quadric surface intersecting at a line.

(ii) The orbit of X is not closed if its equation can be written as $\lambda_1 x_2^3 + \lambda_2 x_1 x_2 x_3 + \lambda_3 x_1 x_2 x_4$ up to a coordinate transform preserving Q. It degenerates to type (ξ).

Otherwise, X is semistable with closed orbit.

Proposition 4.18. *A general member X of type (γ) has two simple elliptic singularity of type $\tilde{E}_8$. Moreover, we have*

(i) X is unstable if X consists of three quadric cones.
(ii) The orbit of X is not closed if its equation has the form $\lambda_1 x_2^3 + \lambda_2 x_1 x_2 x_3 + \lambda_3 x_1^2 x_4$ up to a coordinate change preserving Q.

Otherwise, X is semistable with closed orbit.

5. Complete intersection of three quadrics in $\mathbb{P}^5$

Let $W = H^0(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(2))$ be the space of global sections of $\mathbb{O}(2)$ in $\mathbb{P}^5$. Since every complete intersection X is determined by a net of quadrics Q_1, Q_2, Q_3, the complete intersection of three quadrics are parameterized by the Grassmannian $Gr(3, W)$.

The moduli space of complete intersections can be constructed as the GIT quotient $Gr(3, V_2)//SL_6(\mathbb{C})$ and there is a birational map

$$Gr(3, W)//SL_6(\mathbb{C}) \dashrightarrow \mathcal{M}_8.$$

In this situation, the complete analysis of stable locus is complicated. For example, see [12] for a discussion of GIT stability of a net of quadrics in $\mathbb{P}^4$. However, we are satisfied with the following result:

Theorem 5.1. *Let X be a complete intersection of three quadrics in $\mathbb{P}^5$. If X has simple singularities, then X is GIT stable.*

Before we proceed, we first make some notations. Given a net of quadrics $\{Q_1, Q_2, Q_3\}$, the Plücker coordinates of $\{Q_1, Q_2, Q_3\}$ in $\mathbb{P}(\bigwedge^3 W)$ can be represented by

$$\{x_{i_1} x_{j_1} \wedge x_{i_2} x_{j_2} \wedge x_{i_3} x_{j_3}\}$$

for three distinct pairs (i_k, j_k).

Let $\lambda : \mathbb{C}^* \to SL_6(\mathbb{C})$ be a *normalized* one-parameter subgroup, i.e. $\lambda(t) = \mathrm{diag}(t^{a_0}, t^{a_1} \dots, t^{a_5})$ satisfying $a_0 \geq a_1 \dots \geq a_5$ and $\sum_{i=0}^{5} a_i = 0$. We denote by

$$w_\lambda(x_i x_j) := a_i + a_j$$

the weight of the monomial $x_i x_j$ with respect to λ. The weight of a Plücker coordinate $x_{i_1} x_{j_1} \wedge x_{i_2} x_{j_2} \wedge x_{i_3} x_{j_3}$ with respect to λ is simply $\sum_{k=1}^{3} w_\lambda(x_{i_k} x_{j_k})$.

By the Hilbert-Mumford numerical criterion, a net of quadrics $\{Q_1, Q_2, Q_3\}$ is not properly stable if and only if for a suitable choice of coordinates, there exists a normalized 1-PS $\lambda : t \to \mathrm{diag}(t^{a_0}, t^{a_1} \dots, t^{a_5})$ such that the weight

of all Plücker coordinates of $\{Q_1, Q_2, Q_3\}$ with respect to λ is not positive. We say that $\{Q_1, Q_2, Q_3\}$ is not properly stable with respect to λ.

Given a normalized 1-PS $\lambda : \mathbb{C}^* \to SL_6(\mathbb{C})$, we can define two complete orders on quadratic monomials:

(1) " $>$ " : $x_0^2 > x_0x_1 > \ldots > x_0x_5 > x_1^2 > x_1x_2 > \ldots > x_4x_5 > x_5^2$.
(2) " $>_\lambda$ " : $x_ix_j >_\lambda x_kx_l$ if either $w_\lambda(x_ix_j) > w_\lambda(x_kx_l)$ or $w_\lambda(x_ix_j) = w_\lambda(x_kx_l)$ for a given normalized 1-PS:λ and $x_ix_j > x_kx_l$.

Since the 1-PS $\lambda : \mathbb{C}^* \to SL_6(\mathbb{C})$ is normalized, $x_ix_j >_\lambda x_kx_l$ implies $\max\{i, j\} > \min\{k, l\}$.

We denote by m_i the leading term of Q_i with respect to the order " $>_\lambda$ " and we say that a monomial $x_kx_l \notin Q_i$ if the quadratic polynomial Q_i does not contain x_kx_l term. Moreover, we can always set

$$m_1 >_\lambda m_2 >_\lambda m_3, \tag{5.1}$$

up to replacing Q_1, Q_2, Q_3 with a linear combination of the three polynomials. Then the term $m_1 \wedge m_2 \wedge m_3$ appears in the Plücker coordinates of $Q_1 \wedge Q_2 \wedge Q_3$ and has the largest weight with respect to λ. Hence the net $\{Q_1, Q_2, Q_3\}$ is not properly stable with respect to λ if and only if $w_\lambda(m_1 \wedge m_2 \wedge m_3) \leq 0$.

Lemma 5.2. *With the notation above, let X be the complete intersection $Q_1 \cap Q_2 \cap Q_3$. Then X has a singularity with multiplicity greater than two if one of the following conditions does not hold:*

(1) $m_1 \geq_\lambda x_0x_4$,
(2) $m_2 \geq_\lambda x_1x_5$ if $m_1 = x_0^2$, and $m_2 \geq_\lambda x_0x_5$ otherwise,
(3) $m_3 \geq_\lambda x_3^2$ if $m_1 <_\lambda x_0x_3$.

Moreover, X is singular along a curve if one of the following conditions does not hold:

(1') $m_1 \geq_\lambda x_1^2$ if $m_3 <_\lambda x_1x_5$ or $m_2 <_\lambda x_1x_4$; and $m_1 \geq_\lambda \max\{x_1x_3, x_2^2\}$ otherwise,
(2') $m_2 \geq_\lambda x_2^2$ if $m_3 <_\lambda x_2x_5$; $m_2 \geq \max\{x_1x_4, x_3^2\}$ if $m_1 <_\lambda x_1^2$; and $m_2 \geq_\lambda \max\{x_2x_4, x_3^2\}$ otherwise;
(3') $m_3 \geq \max\{x_3x_5, x_4^2\}$.

Proof. Let p_0 be the point $[1, 0, 0, 0, 0, 0]$ in $\mathbb{P}^5$. For (1) and (2), if either $m_1 <_\lambda x_0x_4$ or $m_2 <_\lambda x_0x_5$ and $m_1 <_\lambda x_0^2$, the surface X contains the point p_0 and two quadrics Q_2, Q_3 are both singular at p_0. It follows that multiplicity of p_0 is greater than 2.

If $m_1 = x_0^2$ and $m_2 <_\lambda x_1x_5$, then X is singular along the two points

$$\{Q_1 = x_2 = x_3 = x_4 = x_5 = 0\}$$

with multiplicity greater than 2. Similarly, one can easily check our assertion for (3).

For (1'), (2') and (3'), we will only list the singular locus of X and leave the proof to readers:

- X is singular along the line $L : x_2 = x_3 = x_4 = x_5 = 0$ if condition $(1')$ is invalid.
- X is either reducible or singular along L or $C_1 : x_3 = x_4 = x_5 = Q_1 = 0$ if condition $(2')$ is invalid.
- X is either reducible or singular along the curve $C_2 : x_4 = x_5 = Q_1 = Q_2 = 0$ if condition $(3')$ is invalid.

♣

As before, we need to know the maximal set $M_{\leq 0}(\lambda)$ of triples of distinct quadratic monomials $\{q_1, q_2, q_3\}$, whose sum of their weights with respect to λ is non-positive. Instead of looking at all maximal subsets, we are interested in the maximal subset $\overline{M}_{\leq 0}(\lambda)$ which contains a triple $\{m_1, m_2, m_3\}$ satisfying the conditions $(1) - (3)$ and $(1') - (3')$ in Lemma 5.2. It is not difficult to compute that there are four such maximal subset. See Table 5 below.

TABLE 3. Maximal set $\overline{M}_{\leq 0}(\lambda)$

Cases	$\lambda = (a_0, \dots, a_5)$	Maximal triples $\{q_1, q_2, q_3\}$		
		q_1	q_2	q_3
$(N1')$	$(2, 1, 0, 0, -1, -2)$	x_0x_2, x_1^2	x_0x_5, x_1x_4, x_2^2	x_2x_5, x_4^2
$(N2')$	$(3, 1, 1, -1, -1, -3)$	x_0x_3, x_1^2	x_0x_5, x_1x_3	x_1x_5, x_3^2
$(N3')$	$(4, 1, 1, -2, -2, -2)$	x_0x_3, x_1^2	x_0x_3, x_1^2	x_3^2
$(N4')$	$(5, 3, 1, -1, -3, -5)$	x_0x_4, x_1x_3, x_2^2	x_0x_5, x_1x_4, x_2x_3	x_1x_5, x_2x_4, x_3^2

The lemma below gives a geometric description of X of type $(N1')-(N4')$.

Lemma 5.3. *Let X be a general element of type $(N1') - (N4')$. Then X has an isolated simple elliptic singularity.*

Proof. Obviously, X is singular at $p_0 = [1, 0, 0, 0, 0, 0]$. Moreover, p_0 is an isolated hypersurface singularity when X is general. To show it is simple elliptic, let us compute the analytic type of p_0 case by case.

If X is a general element of type $(N1')$, then the equations of Q_i can be written as

$$Q_1 : x_0x_2 + q(x_1, \dots, x_5) = 0$$
$$Q_2 : x_0x_5 + x_1x_4 + q'(x_2, x_3) = 0$$
$$Q_3 : x_4^2 + x_5 l(x_2, x_3, x_4, x_5) = 0$$

up to a linear change of the coordinates. Let us take the local coordinates near p_0:

$$y_i = x_i/x_0. \tag{5.2}$$

From the first two quadratic equations, one can get

$$y_2 = f_1(y_1, y_3, y_4),$$
$$y_5 = y_1y_4 + by_3^2 + b'y_3f_1(y_1, y_2, y_4) + f_2(y_1, y_3, y_4),$$

for some formal power series $f_1 \in \mathbb{C}[[y_1, y_3, y_4]]_{\geq 2}, f_2 \in \mathbb{C}[[y_1, y_3, y_4]]_{\geq 4}$ and some constant $b, b' \in \mathbb{C}$. Therefore, the local equation of p_0 is

$$y_4^2 + \alpha_1 y_3^3 + \alpha_2 y_3^2 y_1^2 + \alpha_3 y_3 y_1^4 + \alpha_4 y_1^6 + (\geq \text{higher order terms}) = 0, \tag{5.3}$$

for some complex number α_i. According to §3.1, the singularity p_0 is simple elliptic of type $\tilde{E}_8$.

If X is a general element of type $(N2')$, we write the equations as

$$Q_1 : x_0 x_3 + q(x_1, \dots, x_5) = 0$$
$$Q_2 : x_0 x_5 + x_1 x_3 + x_2 x_4 = 0$$
$$Q_3 : q'(x_3, x_4, x_5) + x_5 l(x_1, x_2) = 0$$

Still, we take the affine coordinate (5.2) near p_0 and then we have

$$y_3 = f(y_1, y_2, y_4),\ y_5 = -y_1 f(y_1, y_2, y_4) - y_2 y_4,$$

for some $f \in \mathbb{C}[[y_1, y_2, y_4]]_{\geq 2}$. Thus the local equation around p_0 is

$$\alpha y_4^2 + g(y_1, y_4) + y_4 g'(y_1, y_2, y_4) = 0. \tag{5.4}$$

where $g \in \mathbb{C}[[y_1, y_2]]_{\geq 4}$, $g' \in \mathbb{C}[[y_1, y_2, y_4]]_{\geq 2}$ and $\alpha \in \mathbb{C}$ is a constant. Hence p_0 is simple elliptic of type $\tilde{E}_7$ by §3.1.

One can similarly prove that X has a simple elliptic singularity p_0 of type $\tilde{E}_7$ when it is general of type $(N3')$, and of type $\tilde{E}_8$ when it is general of type $(N4')$. ♣

Let $\mathcal{U}_8 \subset Gr(3, W)$ be the open subset consisting of all complete intersections with at simplest singularities. Then we can consider $\mathcal{K}_8 = \mathcal{U}_8 // SL_6(\mathbb{C})$ as the moduli space of the complete intersection of three quadrics in $\mathbb{P}^5$ with simplest simplest singularities. Similarly, we can get the following result from Lemma3.3:

Theorem 5.4. *There is an open immersion $\mathcal{P}_8 : \mathcal{K}_8 \to \mathcal{M}_8$ and the complement of $\mathcal{P}_8(\mathcal{K}_8)$ in $\mathcal{M}_8$ is the union of three Noether-Lefschetz divisors $D^8_{1,1}, D^8_{2,1}$ and $D^8_{3,1}$. In particular, the Picard group* $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{M}_8)$ *is spanned by $\{D^8_{d,1},\ 1 \leq d \leq 4\}$.*

We call the Noether-Lefschetz divisors $D^{2l}_{d,1}$ *elliptic* divisors. There is a natural question:

Question 5.5. Are all Noether-Lefschetz divisors supported on elliptic divisors? or equivalently, is the subgroup $\mathrm{Pic}^{NL}_{\mathbb{Q}}(\mathcal{M}_{2l}) \subseteq \mathrm{Pic}_{\mathbb{Q}}(\mathcal{M}_{2l})$ spanned by elliptic divisors $\{D^{2l}_{d,1}, d \in \mathbb{N}\}$?

It remains open when $2l$ is large. This question is related to the problem of coefficients of modular forms. In [25], Maulik has shown that the Hodge line bundle on $\mathcal{M}_{2l}$ is supported on elliptic divisors. His proof relies on the estimate of the coefficients of a vector-valued cusp form (see [25] Lemma 3.7).

6. Cohomology on Shimura varieties

In this section, we discuss the relation between the Picard group $\mathrm{Pic}(\mathcal{M}_{2l})$ and second cohomology group of $\mathcal{M}_{2l}$. Our work is based on the study of various cohomology groups on Shimura varieties associated to orthogonal groups.

6.1. **L^2-cohomology on Shimura variety.** Let $G = SO(2,n)$ be the orthogonal group over $\mathbb{Q}$ and K the maximal compact subgroup of G. Let $\mathcal{D}$ be the Hermitian symmetric space attached to $G(\mathbb{R})$, i.e. $\mathcal{D} = G(\mathbb{R})/K(\mathbb{R})$, and let Γ be an arithmetic subgroup of $G(\mathbb{Q})$.

The arithmetic quotient $X_\Gamma := \Gamma\backslash\mathcal{D}$ is a connected component of the Shimura variety $Sh(G,\mathcal{D})$ (cf. [26] §5). Let $H^k_{(2)}(X_\Gamma,\mathbb{C})$ be the k-th L^2-cohomology of X_Γ. When $H^k_{(2)}(X_\Gamma,\mathbb{C})$ has finite dimension, Hodge theory shows that $H^k_{(2)}(X_\Gamma,\mathbb{C})$ is isomorphic to the space of L^2-harmonic forms which has a natural Hodge structure (cf. [9][14]).

Lemma 6.2. *Suppose X_Γ is smooth and $n > 4$. Then $H^k(X_\Gamma,\mathbb{C})$ has a pure Hodge structure for $k = 1,2$. Moreover, $H^1(X_\Gamma,\mathbb{C}) = H^{2,0}(X_\Gamma) = H^{0,2}(X_\Gamma) = 0$.*

Proof. Let X^*_Γ be the Baily-Borel compactification of X_Γ, whose boundary component has codimension at least 2 (cf. [6][30]).

According to Zucker's conjecture [22] and Durfee's result [11] Prop 3, we get a sequence of isomorphisms

$$H^2_{(2)}(X_\Gamma,\mathbb{C}) \xrightarrow{\sim} IH(X^*_\Gamma,\mathbb{C}) \xrightarrow{\sim} H^2(X_\Gamma,\mathbb{C}) \tag{6.1}$$

where $IH(X^*_\Gamma,\mathbb{C})$ is the second intersection cohomology (with the middle perversity) of X^*_Γ. As shown by Harris and Zucker [14] Thm 5.4, the composition of (6.1) is a (mixed) Hodge morphism. Since the Hodge structure of $H^k_{(2)}(X_\Gamma,\mathbb{C})$ is pure, it follows that $H^k(X_\Gamma,\mathbb{C})$ has a pure Hodge structure for $k \leq 2$.

Next, by Matsushima's formula (e.g. [5]), the L^2-cohomology $H^k_{(2)}(X_\Gamma,\mathbb{C})$ can be expressed as the direct sum of the relative Lie algebra cohomology $H^k(\mathfrak{g},K;\pi)$ (cf. [32]), where $\mathfrak{g}$ is the Lie algebra of $G(\mathbb{R})$ and π is a $(\mathfrak{g},K)$-module. As shown in [13] §1.5 and [3] §5.10, we have

$$H^1(\mathfrak{g},K;\pi) = H^{2,0}(\mathfrak{g},K;\pi) = H^{0,2}(\mathfrak{g},K;\pi) = 0. \tag{6.2}$$

Our assertion follows easily from (6.2). ♣

6.3. **Proof of Theorem 1.2.** In our case, the arithmetic quotient $\mathcal{M}_{2l} = \Gamma_{2l}\backslash\mathcal{D}_{2l}$ is associated to $SO(2,n)$ with some quotient singularities. One can simply choose a torsion-free subgroup $\Gamma'_{2l} \subseteq \Gamma_{2l}$ of finite index. Then $\mathcal{M}'_{2l} := \Gamma'_{2l}\backslash\mathcal{D}_{2l}$ is smooth and

$$H^1(\mathcal{M}'_{2l},\mathcal{O}_{\mathcal{M}'_{2l}}) = H^2(\mathcal{M}'_{2l},\mathcal{O}_{\mathcal{M}'_{2l}}) = 0$$

by Lemma 6.2.

Next, we known that $H := \Gamma'_{2l}\backslash\Gamma_{2l}$ is a finite group and $\mathcal{M}'_{2l} = H\backslash\mathcal{M}_{2l}$. Let $H^k(\mathcal{M}'_{2l}, \mathcal{O}_{\mathcal{M}'_{2l}})^H$ be the H-invariant cohomology class in $H^k(\mathcal{M}'_{2l}, \mathcal{O}_{\mathcal{M}'_{2l}})$, then it is easy to see that

$$H^k(\mathcal{M}_{2l}, \mathcal{O}_{\mathcal{M}_{2l}}) = H^k(\mathcal{M}'_{2l}, \mathcal{O}_{\mathcal{M}'_{2l}})^H = 0.$$

It follows from the exponential exact sequence that $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{M}_{2l}) \cong H^2(\mathcal{M}_{2l}, \mathbb{Q})$

Remark 6.4. The Noether-Lefschetz divisors on $\mathcal{M}_{2l}$ actually corresponds to codimension one subshimura varieties on $\mathcal{M}_{2l}$, which are called special cycles on a Shimura variety associated to an orthogonal group. Then one can easily see that to prove the Noether-Lefschetz conjecture on $\mathcal{M}_{2l}$, it suffices to show that the second cohomology group of $\mathcal{M}'_{2l}$ is spanned by special cycles for some arithmetic subgroup $\Gamma'_{2l} \subset \Gamma_{2l}$.

References

[1] V. I. Arnold, S. M. Gusein-Zade, and A. N. Varchenko. *Singularities of differentiable maps. Volume 1.* Modern Birkhäuser Classics. Birkhäuser/Springer, New York, 2012.

[2] Olivier Benoist. Quelques espaces de modules d'intersections compltes lisses qui sont quasi-projectifs. *arXiv:1111.1589*, 2011.

[3] N. Bergeron, J. Millson, and C. Moeglin. Hodge type theorems for arithmetic manifolds associated to orthogonal groups. *arXiv:1110.3049*, 2011.

[4] Bruce C. Berndt and Ronald J. Evans. The determination of Gauss sums. *Bull. Amer. Math. Soc. (N.S.)*, 5(2):107–129, 1981.

[5] A. Borel and N. Wallach. *Continuous cohomology, discrete subgroups, and representations of reductive groups*, volume 67 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, second edition, 2000.

[6] Armand Borel and Lizhen Ji. *Compactifications of symmetric and locally symmetric spaces*. Mathematics: Theory & Applications. Birkhäuser Boston Inc., Boston, MA, 2006.

[7] Jan Hendrik Bruinier. On the rank of Picard groups of modular varieties attached to orthogonal groups. *Compositio Math.*, 133(1):49–63, 2002.

[8] Sebastian Casalaina-Martin, David Jensen, and Radu Laza. Log canonical models and variation of git for genus four canonical curves. *arXiv:1203.5014*, 2012.

[9] Xianzhe Dai. An introduction to L^2 cohomology. In *Topology of stratified spaces*, volume 58 of *Math. Sci. Res. Inst. Publ.*, pages 1–12. Cambridge Univ. Press, Cambridge, 2011.

[10] Morrison David. The geometry of K3 surfaces. *Lectures at the Scuola mathematic interuniversitaria corona, Italy*, 1988.

[11] Alan H. Durfee. Intersection homology Betti numbers. *Proc. Amer. Math. Soc.*, 123(4):989–993, 1995.

[12] Maksym Fedorchuk. The final log canonical model of the moduli space of stable curves of genus four. *arXiv:1106.5012v2*, 2011.

[13] Michael Harris and Jian-Shu Li. A Lefschetz property for subvarieties of Shimura varieties. *J. Algebraic Geom.*, 7(1):77–122, 1998.

[14] Michael Harris and Steven Zucker. Boundary cohomology of Shimura varieties. III. Coherent cohomology on higher-rank boundary strata and applications to Hodge theory. *Mém. Soc. Math. Fr. (N.S.)*, (85):vi+116, 2001.

[15] Brendan Hassett. Special cubic fourfolds. *Compositio Math.*, 120(1):1–23, 2000.

[16] Trygve Johnsen and Andreas Leopold Knutsen. *K3 projective models in scrolls*, volume 1842 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2004.

[17] Frances Kirwan. Moduli spaces of degree d hypersurfaces in $\mathbf{P}_n$. *Duke Math. J.*, 58(1):39–78, 1989.

[18] Frances Kirwan and Ronnie Lee. The cohomology of moduli spaces of $K3$ surfaces of degree 2. I. *Topology*, 28(4):495–516, 1989.

[19] Friedrich Knop, Hanspeter Kraft, and Thierry Vust. The Picard group of a G-variety. In *Algebraische Transformationsgruppen und Invariantentheorie*, volume 13 of *DMV Sem.*, pages 77–87. Birkhäuser, Basel, 1989.

[20] Shigeyuki Kondō. On the Albanese variety of the moduli space of polarized $K3$ surfaces. *Invent. Math.*, 91(3):587–593, 1988.

[21] Radu Laza. The moduli space of cubic fourfolds. *J. Algebraic Geom.*, 18(3):511–545, 2009.

[22] Eduard Looijenga. L^2-cohomology of locally symmetric varieties. *Compositio Math.*, 67(1):3–20, 1988.

[23] D. Luna. Adhérences d'orbite et invariants. *Invent. Math.*, 29(3):231–238, 1975.

[24] D. Maulik and R. Pandharipande. Gromov-Witten theory and Noether-Lefschetz theory. *arXiv:0705.1653*, 2007.

[25] Davesh Maulik. Supersingular K3 surfaces for large primes. *arXiv:1203.2889*, 2012.

[26] J. S. Milne. Introduction to Shimura varieties. In *Harmonic analysis, the trace formula, and Shimura varieties*, volume 4 of *Clay Math. Proc.*, pages 265–378. Amer. Math. Soc., Providence, RI, 2005.

[27] D. Mumford, J. Fogarty, and F. Kirwan. *Geometric invariant theory*, volume 34 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)]*. Springer-Verlag, Berlin, third edition, 1994.

[28] V. V. Nikulin. Integer symmetric bilinear forms and some of their geometric applications. *Izv. Akad. Nauk SSSR Ser. Mat.*, 43(1):111–177, 238, 1979.

[29] B. Saint-Donat. Projective models of K-3 surfaces. *Amer. J. Math.*, 96:602–639, 1974.

[30] Francesco Scattone. On the compactification of moduli spaces for algebraic $K3$ surfaces. *Mem. Amer. Math. Soc.*, 70(374):x+86, 1987.

[31] Jayant Shah. Degenerations of $K3$ surfaces of degree 4. *Trans. Amer. Math. Soc.*, 263(2):271–308, 1981.

[32] David A. Vogan, Jr. and Gregg J. Zuckerman. Unitary representations with nonzero cohomology. *Compositio Math.*, 53(1):51–90, 1984.

Department of Mathematics, building 380, Stanford, CA 94305, U.S.A.

Department of Mathematics, California Institute of Technology, Pasadena, CA, 91125, U.S.A.

E-mail address: zli2@stanford.edu, tian@caltech.edu